# Generic section of a hyperplane arrangement and twisted Hurewicz maps

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February 2, 2008

**Abstract**

We consider a twisted version of the Hurewicz map on the complement of a hyperplane arrangement. The purpose of this paper is to prove surjectivity of the twisted Hurewicz map under some genericity conditions. As a corollary, we also prove that a generic section of the complement of a hyperplane arrangement has non-trivial homotopy groups.

## 1 Twisted Hurewicz map

Let $X$ be a topological space with a base point $x_0 \in X$ and $\mathcal{L}$ a local system of $\mathbb{Z}$-modules on $X$. Let $f : (S^n, *) \to (X, x_0)$ be a continuous map from the sphere $S^n$ with $n \geq 2$. Since $S^n$ is simply connected, the pullback $f^*\mathcal{L}$ turns out to be a trivial local system. Thus given a local section $t \in \mathcal{L}_{x_0}$, $f \otimes t$ determines a twisted cycle with coefficients in $\mathcal{L}$. This induces a twisted version of the Hurewicz map:

$$h : \pi_n(X, x_0) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0} \longrightarrow H_n(X, \mathcal{L}).$$

The classical Hurewicz map is corresponding to the case of trivial local system $\mathcal{L} = \mathbb{Z}$ with $t = 1$.

## 2 Main result

Let $\mathcal{A}$ be an essential affine hyperplane arrangement in an affine space $V = \mathbb{C}^\ell$, with $\ell \geq 3$. Let $M(\mathcal{A})$ denote the complement $V - \bigcup_{H\in\mathcal{A}} H$. A hyperplane $U \subset V$ is said to be *generic to* $\mathcal{A}$ if $U$ is transversal to the stratification induced from $\mathcal{A}$. Let $i : U \cap M(\mathcal{A}) \hookrightarrow M(\mathcal{A})$ denote the inclusion.

In this notation, the main result of this paper is the following:

**Theorem 1** *Let $\mathcal{L}' := i^*\mathcal{L}$ be the restriction of a nonresonant local system $\mathcal{L}$ of arbitrary rank on $M(\mathcal{A})$. Then the twisted Hurewicz map*

$$h : \pi_{\ell-1}(U \cap M(\mathcal{A}), x_0) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0} \longrightarrow H_{\ell-1}(U \cap M(\mathcal{A}), \mathcal{L}')$$

*is surjective.*

For the notion of "nonresonant local system", see Theorem 7.

Theorem 1 should be compared with a result proved by Randell in [12]. He proved that the Hurewicz homomorphism $\pi_k(M(\mathcal{A})) \to H_k(M(\mathcal{A}), \mathbb{Z})$ is equal to the zero map when $k \geq 2$ for any $\mathcal{A}$. However little is known about twisted Hurewicz maps for other cases.

The key ingredient for our proof of Theorem 1 is an affine Lefschetz theorem of Hamm, which asserts that $M(\mathcal{A})$ has the homotopy type of a finite CW complex whose $(\ell-1)$-skeleton has the homotopy type of $U \cap M(\mathcal{A})$. We obtain $(\ell-1)$-dimensional spheres in $U \cap M(\mathcal{A})$ as boundaries of the $\ell$-dimensional top cells. Applying a vanishing theorem for local system homology groups, we show that these spheres generate the twisted homology group $H_{\ell-1}(U \cap M(\mathcal{A}), \mathcal{L})$. We should note that the essentially same arguments are used in [4] to compute the rank of $\pi_{\ell-1}(U \cap M(\mathcal{A}), x_0) \otimes_{\mathbb{Z}} \mathcal{L}_{x_0}$ under a certain asphericity condition on $\mathcal{A}$.

## 3 Topology of complements

The cell decompositions of affine varieties or hypersurface complements are well studied subjects. Let $f \in \mathbb{C}[x_1, \cdots, x_\ell]$ be a polynomial and $D(f) := \{x \in \mathbb{C}^\ell |\ f(x) \neq 0\}$ be the hypersurface complement defined by $f$.

**Theorem 2** (**Affine Lefschetz Theorem [5]**) Let $U$ be a sufficiently generic hyperplane in $\mathbb{C}^\ell$. Then,

(a) The space $D(f)$ has the homotopy type of a space obtained from $D(f)\cap U$ by attaching $\ell$-dimensional cells.

(b) Let $i_p : H_p(D(f)\cap U,\mathbb{Z}) \longrightarrow H_p(D(f),\mathbb{Z})$ denote the homomorphism induced by the natural inclusion $i : D(f)\cap U \hookrightarrow D(f)$. Then

$$i_p \text{ is } \begin{cases} \text{isomorphic} & \text{for } p = 0,1,\cdots,\ell-2 \\ \text{surjective} & \text{for } p = \ell-1. \end{cases}$$

Suppose $i_{\ell-1}$ is also isomorphic. Then as noted by Dimca and Papadima [3] (see also Randell [13]), the number of $\ell$-dimensional cells attached would be equal to the Betti number $b_\ell(D(f))$ and the chain boundary map $\partial : C_\ell(D(f),\mathbb{Z}) \to C_{\ell-1}(D(f),\mathbb{Z})$ of the cellular chain complex associated to the cell decomposition is equal to zero. Otherwise $i_{\ell-1} : H_{\ell-1}(D(f)\cap U,\mathbb{Z}) \longrightarrow H_{\ell-1}(D(f),\mathbb{Z})$ has a nontrivial kernel $\partial(C_\ell(D(f),\mathbb{Z}))$.

In the case of hyperplane arrangements, homology groups and homomorphisms $i_p$ are described combinatorially in terms of the intersection poset [9]. Let us recall some notation. Let $\mathcal{A}$ be a finite set of affine hyperplanes in $\mathbb{C}^\ell$,

$$L(\mathcal{A}) = \{X = \bigcap_{H\in I} H \mid I \subset \mathcal{A}\}$$

be the set of nonempty intersections of elements of $\mathcal{A}$ with reverse inclusion $X < Y \Longleftrightarrow X \supset Y$, for $X,Y \in L(\mathcal{A})$. Define a rank function on $L(\mathcal{A})$ by

$$r : L(\mathcal{A}) \longrightarrow \mathbb{Z}_{\geq 0},\ X \longmapsto \mathrm{codim} X,$$

the Möbius function $\mu : L(\mathcal{A}) \longrightarrow \mathbb{Z}$ by

$$\mu(X) = \begin{cases} 1 & \text{for } X = V \\ -\sum_{Y<X}\mu(Y), & \text{for } X > V, \end{cases}$$

and the characteristic polynomial $\chi(\mathcal{A},t)$ by

$$\chi(\mathcal{A},t) = \sum_{X\in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

Let $E^1 = \bigoplus_{H\in\mathcal{A}} \mathbb{C}e_H$ and $E = \wedge E^1$ be the exterior algebra of $E^1$, with $p$-th graded term $E^p = \overset{p}{\wedge} E^1$. Define a $\mathbb{C}$-linear map $\partial : E \to E$ by $\partial 1 = 0$, $\partial e_H = 1$ and for $p \geq 2$

$$\partial(e_{H_1}\cdots e_{H_p}) = \sum_{k=1}^{p} (-1)^{k-1} e_{H_1}\cdots \widehat{e_{H_k}} \cdots e_{H_p}$$

for all $H_1, \cdots, H_p \in \mathcal{A}$. A subset $S \subset \mathcal{A}$ is said to be dependent if $r(\cap S) < |S|$, where $\cap S = \cap_{H \in S} H$. For $S = \{H_1, \cdots, H_p\}$, we write $e_S := e_{H_1} \cdots e_{H_p}$.

**Definition 3** Let $I(\mathcal{A})$ be the ideal of $E(\mathcal{A})$ generated by

$$\{e_S \mid \cap S = \phi\} \cup \{\partial e_S \mid S \text{ is dependent}\}.$$

The Orlik-Solomon algebra $A(\mathcal{A})$ is defined by $A(\mathcal{A}) = E(\mathcal{A})/I(\mathcal{A})$.

**Theorem 4** (**Orlik-Solomon** [8]) Fix a defining linear form $\alpha_H$ for each $H \in \mathcal{A}$. Then the correspondence $e_H \mapsto d \log \alpha_H$ induces an isomorphism of graded algebras:

$$A(\mathcal{A}) \xrightarrow{\cong} H^*(M(\mathcal{A}), \mathbb{C}).$$

The Betti numbers of $M(\mathcal{A})$ are given by

$$\chi(\mathcal{A}, t) = \sum_{k=0}^{\ell} (-1)^k b_k(M(\mathcal{A})) t^{\ell-k}.$$

From the above description of cohomology ring of $M(\mathcal{A})$, we have:

**Theorem 5** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$ and $U$ be a hyperplane generic to $\mathcal{A}$. Then $i : U \cap M(\mathcal{A}) \hookrightarrow M(\mathcal{A})$ induces isomorphisms $i_p : H_p(M(\mathcal{A}) \cap U, \mathbb{Z}) \xrightarrow{\cong} H_p(M(\mathcal{A}), \mathbb{Z})$ for $p = 0, \cdots, \ell - 1$.

**Proof.** It is easily seen from the genericity that

$$L(\mathcal{A} \cap U) \cong L_{\leq \ell-1}(\mathcal{A}) := \{X \in L(\mathcal{A}) |\ r(X) \leq \ell - 1\}. \tag{1}$$

In particular a generic intersection preserves the part of rank $\leq \ell - 1$. Hence $A(\mathcal{A} \cap U) \cong A^{\leq \ell-1}(\mathcal{A})$. This induces isomorphisms $H^{\leq \ell-1}(M(\mathcal{A})) \cong H^{\leq \ell-1}(M(\mathcal{A}) \cap U)$. Since homology groups $H_*(M(\mathcal{A}), \mathbb{Z})$ are torsion free, the theorem is the dual of these isomorphisms. □

Using these results inductively, the complement $M(\mathcal{A})$ of the hyperplane arrangement $\mathcal{A}$ has a minimal cell decomposition.

**Theorem 6** ([**13**][**3**][**11**]) The complement $M(\mathcal{A})$ is homotopic to a minimal CW cell complex, i.e. the number of $k$-dimensional cells is equal to the Betti number $b_k(M(\mathcal{A}))$ for each $k = 0, \cdots, \ell$.

## 4 Proof of the main theorem

First we recall the vanishing theorem of homology groups for a "generic" or nonresonant local system $\mathcal{L}$ of complex rank $r$.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^\ell$, let $U$ be a hyperplane generic to $\mathcal{A}$ and let $i : M(\mathcal{A}) \cap U \hookrightarrow M(\mathcal{A})$ be the inclusion. Now we assume that $\mathcal{A}$ is essential, i.e., $\mathcal{A}$ contains linearly independent $\ell$ hyperplanes $H_1, \cdots, H_\ell \in \mathcal{A}$.

Let $\mathbb{P}^\ell$ be the projective space, which is a compactification of our vector space $V$. The projective closure of $\mathcal{A}$ is defined as $\mathcal{A}_\infty := \{\bar{H} | \ H \in \mathcal{A}\} \cup \{H_\infty\}$, where $\mathbb{P}^\ell = V \cup H_\infty$. A non-empty intersection $X \in L(\mathcal{A}_\infty)$ defines the subarrangement $(\mathcal{A}_\infty)_X = \{H \in \mathcal{A}_\infty | \ X \subset H\}$ of $\mathcal{A}_\infty$. A subspace $X \in \mathcal{A}_\infty$ is called dense if $(\mathcal{A}_\infty)_X$ is indecomposable, that is, not the product of two non-empty arrangements. Let $\rho : \pi_1(M(\mathcal{A}), x_0) \to GL_r(\mathbb{C})$ be the monodromy representation associated to $\mathcal{L}$. Choosing a point $p \in X \setminus \bigcup_{H \in \mathcal{A}_\infty \setminus (\mathcal{A}_\infty)_X} H$ and a generic line $L$ passing through $p$. Then the small loop $\gamma$ on $L$ around $p \in L$ determines a total turn monodromy $\rho(\gamma) \in GL_r(\mathbb{C})$. The conjugacy class of $\rho(\gamma)$ in $GL_r(\mathbb{C})$ depends only on $X \in L(\mathcal{A}_\infty)$, which is denoted by $T_X$.

The following vanishing theorem of local system cohomology groups is obtained in [2].

**Theorem 7** Let $\mathcal{L}$ be a nonresonant local system on $M(\mathcal{A})$ of rank $r$, that is, for each dense subspace $X \subset H_\infty$ the corresponding monodromy operator $T_X$ does not admit 1 as an eigenvalue. Then

$$\dim H^k(M(\mathcal{A}), \mathcal{L}) = \begin{cases} (-1)^\ell r \cdot \chi(M(\mathcal{A})) & \text{for } k = \ell \\ 0 & \text{for } k \neq \ell, \end{cases}$$

where $\chi(M(\mathcal{A}))$ is the Euler characteristic of the space $M(\mathcal{A})$.

Note that $\mathcal{L}$ is nonresonant if and only if the dual local system $\mathcal{L}^\vee$ is nonresonant. From the universal coefficient theorem

$$H^k(M(\mathcal{A}), \mathcal{L}) \cong \mathrm{Hom}_\mathbb{C}(H_k(M(\mathcal{A}), \mathcal{L}^\vee), \mathbb{C}),$$

we also have the similar vanishing theorem for local system homology groups $H_k(M(\mathcal{A}), \mathcal{L})$.

From Theorem 2 (a) we may identify, up to homotopy equivalence, $M(\mathcal{A})$ with a finite $\ell$-dimensional CW complex for which the

$$(\ell - 1)\text{-skeleton has the homotopy type of } M(\mathcal{A}) \cap U. \qquad (2)$$

We denote the attaching maps of $\ell$-cells by $\phi_k : \partial c_k \cong \mathrm{S}^{\ell-1} \to M(\mathcal{A}) \cap U$, $(k = 1, \cdots, b = b_\ell(M(\mathcal{A})))$, where $c_k \cong \mathrm{D}^\ell$ is the $\ell$-dimensional unit disk. Hence $\phi = \{\phi_k\}_{k=1,\cdots,b}$ satisfies

$$\left( (M(\mathcal{A}) \cap U) \cup_\phi \bigcup_k c_k \right) \text{ is homotopic to } M(\mathcal{A}).$$

Let $\mathcal{L}$ be a rank $r$ local system over $M = M(\mathcal{A})$. For our purposes, it suffices to prove that $h(\phi_k)$ $(k = 1, \cdots, b)$ generate $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L})$.

Let

$$0 \longrightarrow C_\ell \xrightarrow{\partial_{\mathcal{L}}} C_{\ell-1} \xrightarrow{\partial_{\mathcal{L}}} \cdots \xrightarrow{\partial_{\mathcal{L}}} C_0 \longrightarrow 0 \tag{3}$$

be the twisted cellular chain complex associated with the CW decomposition for $M(\mathcal{A})$. Then from (2), the twisted chain complex for $M(\mathcal{A}) \cap U$ is obtained by truncating (3) as

$$0 \longrightarrow C_{\ell-1} \xrightarrow{\partial_{\mathcal{L}}} \cdots \xrightarrow{\partial_{\mathcal{L}}} C_0 \longrightarrow 0. \tag{4}$$

It is easily seen that if $\mathcal{L}$ is generic in the sense of Theorem 7, then the restriction $i^*\mathcal{L}$ is also generic. Applying Theorem 7 to (3), only the $\ell$-th homology survives. Similarly, only the $(\ell-1)$-st homology survives in (4). Note that $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L}) = \mathrm{Ker}(\partial_{\mathcal{L}} : C_{\ell-1} \to C_{\ell-2})$. Thus we conclude that

$$\partial_{\mathcal{L}} : C_\ell \longrightarrow H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L}) \tag{5}$$

is surjective. Since the map (5) is determined by

$$C_\ell \ni [c_k] \longmapsto [\partial c_k] = h(\phi_k),$$

$\{h(\phi_k)\}_{k=1,\cdots,b}$ generate $H_{\ell-1}(M(\mathcal{A}) \cap U, i^*\mathcal{L})$. This completes the proof of Theorem 1.

**Lemma 8** The Euler characteristic of $M(\mathcal{A}) \cap U$ is not equal to zero, more precisely,

$$(-1)^{\ell-1}\chi(M(\mathcal{A}) \cap U) > 0.$$

Given a hyperplane $H \in \mathcal{A}$, we define $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ and $\mathcal{A}'' = \mathcal{A}' \cap H$. Then characteristic polynomials for these arrangements satisfy an inductive formula:

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t).$$

By Theorem 4, the Euler characteristic $\chi(M(\mathcal{A}))$ of the complement is equal to $\chi(\mathcal{A}, 1)$.

**Proof of the Lemma 8.** From (1) and definition of the characteristic polynomial, we have

$$\chi(\mathcal{A} \cap U, t) = \frac{\chi(\mathcal{A}, t) - \chi(\mathcal{A}, 0)}{t}.$$

The proof of the lemma is by induction on the number of hyperplanes. If $|\mathcal{A}| = \ell$, $\mathcal{A}$ is linearly isomorphic to the Boolean arrangement, i.e. one defined by $\{x_1 \cdot x_2 \cdots x_\ell = 0\}$, for a certain coordinate system $(x_1, \cdots, x_\ell)$. In this case, $\chi(\mathcal{A}, t) = (t-1)^\ell$, and we have $(-1)^{\ell-1}\chi(M(\mathcal{A}) \cap U) = 1$. Assume that $\mathcal{A}$ contains more than $\ell$ hyperplanes. We can choose a hyperplane $H \in \mathcal{A}$ such that $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is essential. Then $\mathcal{A}'' = \mathcal{A}' \cap H$ is also essential, and obviously $U$ is generic to $\mathcal{A}'$ and $\mathcal{A}''$. Thus we have

$$\begin{aligned}
(-1)^{\ell-1}\chi(\mathcal{A} \cap U) &= (-1)^{\ell-1}\chi(\mathcal{A} \cap U, 1) \\
&= (-1)^{\ell-1}\left(\chi(\mathcal{A}' \cap U, 1) - \chi(\mathcal{A}'' \cap U, 1)\right) \\
&= (-1)^{\ell-1}\chi(\mathcal{A}' \cap U, 1) + (-1)^{\ell-2}\chi(\mathcal{A}'' \cap U, 1) \\
&> 0.
\end{aligned}$$

$\square$

Using Lemma 8, we have the following non-vanishing of the homotopy group, which generalizes a classical result of Hattori [6].

**Corollary 9** Let $2 \leq k \leq \ell - 1$ and $F^k \subset V$ be a $k$-dimensional subspace generic to $\mathcal{A}$. Then $\pi_k(M(\mathcal{A}) \cap F^k) \neq 0$.

**Remark 10** We can also prove Corollary 9 directly in the following way. Suppose $\pi_{\ell-1}(M(\mathcal{A}) \cap U) = 0$. Then the attaching maps $\{\phi_k : \partial c_k = S^{\ell-1} \to M(\mathcal{A}) \cap U\}$ of the top cells are homotopic to the constant map. Hence we have a homotopy equivalence

$$M(\mathcal{A}) \text{ is homotopic to } (M(\mathcal{A}) \cap U) \vee \bigvee_k S^\ell.$$

However this contradicts to the fact that cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ is generated by degree one elements (Theorem 4). Hence we have $\pi_{\ell-1}(M(\mathcal{A}) \cap U) \neq 0$.

**Remark 11** We should also note that other results on the non-vanishing of higher homotopy groups of generic sections are found in Randell [12] (for generic sections of aspherical arrangements), in Papadima-Suciu [11] (for hypersolvable arrangements) and in Dimca-Papadima [3] (for iterated generic hyperplane sections).

**Acknowledgements.** The author expresses deep gratitude to Professor M. Falk and Professor S. Papadima, for indicating Remark 10 to him. The author also thanks to the referee for a lot of suggestions which improve the paper. The author was supported by the JSPS Postdoctoral Fellowships for Research Abroad.

# References

[1] K. Cho, A generalization of Kita and Noumi's vanishing theorems of cohomology groups of local system. Nagoya Math. J. **147** (1997), 63–69.

[2] D. C. Cohen, A. Dimca, P. Orlik, Nonresonance conditions for arrangements. Ann. Inst. Fourier (Grenoble) **53** (2003), no. 6, 1883–1896.

[3] A. Dimca, S. Papadima, Hypersurface complements Milnor fibers and minimality of arrangements, Ann. Math. **158**(2003), 473–507

[4] A. Dimca, S. Papadima, Equivariant chain complexes, twisted homology and relative minimality of arrangements. Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 3, 449–467.

[5] H. Hamm, Lefschetz theorems for singular varieties, Proc. Symp. Pure Math., Singularities, Volume 40, Part 1(1983), 547–557

[6] A. Hattori, Topology of $C^n$ minus a finite number of affine hyperplanes in general position, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), no. 2, 205–219

[7] T. Kohno, Homology of a local system on the complement of hyperplanes, Proc. Japan Acad. Ser. A **62**(1986), 144–147

[8] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. **56**(1980), 167–189

[9] P. Orlik, H. Terao, Arrangements of Hyperplanes, Springer Verlag, 1992

[10] P. Orlik, H. Terao, Arrangements and Hypergeometric Integrals. MSJ Memoirs **9**, (2001)

[11] S. Papadima, A. Suciu, Higher homotopy groups of complements of complex hyperplane arrangements. Adv. Math. **165** (2002), no. 1, 71–100.

[12] R. Randell, Homotopy and group cohomology of arrangements, Topology Appl. **78** (1997), 201–213

[13] R. Randell, Morse theory, Milnor fibers and minimality of hyperplane arrangements, Proc. Amer. Math. Soc. **130**(2002), 2737–2743